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On residual properties of word hyperbolic groups

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Abstract. For a fixed word hyperbolic group we compare different residual properties related to quasiconvex subgroups.

1 Introduction

Any group G can be equipped with a *profinite topology* $\mathscr{PT}(G)$, whose basic open sets are cosets to normal finite index subgroups. It is easy to see that the group operations are continuous in $\mathscr{PT}(G)$. The group is residually finite if and only if the profinite topology is Hausdorff.

A subgroup $H \leqslant G$ is closed in $\mathscr{PT}(G)$ if and only if it is an intersection of finite index subgroups; equivalently, for any element $g \notin H$ there exists a homomorphism φ from G to a finite group L such that $\varphi(g) \notin \varphi(H)$. In this case the subgroup H is called G-*separable*.

The profinite closure of a subgroup $H \leqslant G$, i.e., the smallest closed subset containing H, is equal to the intersection of all finite index subgroups K of G such that $H \leqslant K$.

A group G is said to be LERF if every finitely generated subgroup is closed in $\mathscr{PT}(G)$. The class of all LERF groups includes free groups [6], surface groups [20] and fundamental groups of certain 3-manifolds [20], [3].

Let G be a (word) hyperbolic group with a finite symmetrized generating set $\mathscr{A}$ and let $\Gamma(G, \mathscr{A})$ be the corresponding Cayley graph of G. A subset $Q \subseteq G$ is said to be *quasiconvex* if there exists a constant $\eta \geqslant 0$ such that for any pair of elements $u, v \in Q$ and any geodesic segment p connecting u and v, p belongs to a closed η-neighborhood of the subset Q in $\Gamma(G, \mathscr{A})$. Quasiconvex subgroups are precisely the finitely generated subgroups which are embedded in G without distortion [10, Lemma 1.6].

As noted in [13], in the context of word hyperbolic groups instead of studying LERF-groups it makes sense to study GFERF-groups. A hyperbolic group is called

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GFERF if each quasiconvex subgroup is closed in $\mathcal{PT}(G)$. Thus every LERF hyperbolic group is GFERF but not vice versa.

Unfortunately, it is completely unclear how to decide if a random word hyperbolic group is GFERF (or LERF). With this purpose, Long [8] and, later, Niblo and Williams [16] suggested utilizing the *engulfing* property. They say that a subgroup $H \leqslant G$ is *engulfed* if it is contained in a proper finite index subgroup of G. The following two theorems were established by Niblo and Williams in 2002:

Theorem A ([16, Theorem 4.1]). *Let G be a word hyperbolic group and suppose that G engulfs every finitely generated free subgroup with limit set a proper subset of the boundary of G. Then the intersection of all finite index subgroups of G is finite. If G is torsion-free then it is residually finite.*

Theorem B ([16, Theorem 5.2]). *Let G be a word hyperbolic group which engulfs every finitely generated subgroup K such that the limit set $\Lambda(K)$ is a proper subset of the boundary of G. Then every quasiconvex subgroup of G has a finite index in its profinite closure in G.*

The main goal of this paper is to generalize Theorems A and B by weakening their assumptions and, in certain situations, strengthening their conclusions.

In a hyperbolic group G, the structure of a distorted subgroup can be very complicated. Thus the basic idea is to use assumptions which concern only quasiconvex subgroups. We prove the following results in Section 5:

Theorem 1. *Let G be a hyperbolic group with a generating set of cardinality $s \in \mathbb{N}$. Suppose that each proper free quasiconvex subgroup of rank s is engulfed in G. Then the intersection of all finite index subgroups of G is finite. If G is torsion-free then it is residually finite.*

Theorem 2. *Suppose that G is a hyperbolic group which engulfs each proper quasiconvex subgroup. Let H be an arbitrary quasiconvex subgroup of G. Then H has finite index in its profinite closure K. Moreover $K \leqslant HQ$, where Q is the intersection of all finite index subgroups of G.*

The assumptions of Theorems 1 and 2 are less restrictive than the assumptions of Theorems A and B, because if H is a quasiconvex subgroup of a hyperbolic group G with $|G : H| = \infty$ then the limit set $\Lambda(H)$ is a proper subset of ∂G (see [22, Theorem 4], [14, Lemma 8.2]).

In the residually finite case, Theorem 2 can be reformulated as follows:

Theorem 3. *Let G be a residually finite hyperbolic group where every proper quasiconvex subgroup is engulfed. Then G is GFERF.*

Combining together the claims of Theorems 1 and 3 one obtains

Corollary 1. *Let G be a torsion-free hyperbolic group where each proper quasiconvex subgroup is engulfed. Then G is GFERF.*

Romanovskiĭ [19] and, independently, Burns [2] showed that a free product of two LERF groups is again a LERF group. We give yet another construction for GFERF groups by proving the corresponding result for them (see Section 6):

Theorem 4. *Suppose that G_1 and G_2 are GFERF hyperbolic groups. Then the free product $G = G_1 * G_2$ is also a GFERF hyperbolic group.*

2 Preliminaries

Let G be a group with a finite symmetrized generating set $\mathscr{A}$. This generating set gives rise to a *word length* $|g|_G$ for every element $g \in G$. The (left-invariant) *word metric* $d : G \times G \to \mathbb{N} \cup \{0\}$ is defined by the formula $d(x, y) = |x^{-1}y|_G$ for any $x, y \in G$. This metric can be canonically extended to the Cayley graph $\Gamma(G, \mathscr{A})$ by making each edge isometric to the interval $[0, 1] \subset \mathbb{R}$.

For any three points $x, y, w \in \Gamma(G, \mathscr{A})$, the *Gromov product* of x and y with respect to w is defined as

$$(x|y)_w := \frac{1}{2}(d(x, w) + d(y, w) - d(x, y)).$$

Since the metric is left-invariant, for arbitrary $x, y, w \in G$ we have

$$(x|y)_w = (w^{-1}x \mid w^{-1}y)_{1_G}.$$

The group G is called (*word*) *hyperbolic* in the sense of Gromov [5] if there exists $\delta \geqslant 0$ such that for any $x, y, z, w \in \Gamma(G, \mathscr{A})$ their Gromov products satisfy

$$(x|y)_w \geqslant \min\{(x|z)_w, (y|z)_w\} - \delta.$$

Equivalently, G is hyperbolic if there exists $\delta \geqslant 0$ such that each geodesic triangle Δ in $\Gamma(G, \mathscr{A})$ is δ-slim, i.e., every side of Δ is contained an a δ-neighborhood of the two other sides (see [1]).

From now on we assume that G is a hyperbolic group and that δ is large enough so that it satisfies the conditions of both of the above definitions.

For any two points $x, y \in \Gamma(G, \mathscr{A})$ we fix a geodesic path between them and denote it by $[x, y]$. Let p be a path in the Cayley graph of G. Then p_-, p_+ will denote the initial point and end-point of p, and $\|p\|$ the length of p; as usual, $\mathrm{lab}(p)$ denotes the word in the alphabet $\mathscr{A}$ written on p. We write $\mathrm{elem}(p) \in G$ for the element of G represented by $\mathrm{lab}(p)$. If W is a word in $\mathscr{A}$, then $\mathrm{elem}(W)$ will denote the corresponding element of G. The closed ε-neighborhood of a subset $A \subset \Gamma(G, \mathscr{A})$ will be denoted by $\mathcal{O}_\varepsilon(A)$.

The δ-slimness of geodesic triangles implies the 2δ-slimness of all geodesic quadrangles $abcd$ in $\Gamma(G, \mathscr{A})$:

$$[a, b] \subset \mathcal{O}_{2\delta}([b, c] \cup [c, d] \cup [a, d]).$$

A path q is called (λ, c)-*quasi-geodesic* if there exist $0 < \lambda \leqslant 1$, $c \geqslant 0$, such that for any subpath p of q the inequality $\lambda\|p\| - c \leqslant d(p_-, p_+)$ holds. A word W is said to be (λ, c)-quasi-geodesic if some (equivalently, every) path q in $\Gamma(G, \mathscr{A})$ labelled by W is (λ, c)-quasi-geodesic.

Lemma 2.1 ([4, (5.6), (5.11)], [1, (3.3)]). *There is a constant $v = v(\delta, \lambda, c)$ such that for any (λ, c)-quasi-geodesic path p in $\Gamma(G, \mathscr{A})$ and a geodesic q with $p_- = q_-$, $p_+ = q_+$, one has $p \subset \mathcal{O}_v(q)$ and $q \subset \mathcal{O}_v(p)$.*

Lemma 2.2 ([14, Lemma 4.1]). *Consider a geodesic quadrangle $x_1x_2x_3x_4$ in the Cayley graph $\Gamma(G, \mathscr{A})$ with $d(x_2, x_3) > d(x_1, x_2) + d(x_3, x_4)$. Then there are points $u, v \in [x_2, x_3]$ such that $d(x_2, u) \leqslant d(x_1, x_2)$, $d(v, x_3) \leqslant d(x_3, x_4)$ and the geodesic subsegment $[u, v]$ of $[x_2, x_3]$ lies 2δ-close to the side $[x_1, x_4]$.*

If $x, g \in G$, we write $x^g = gxg^{-1}$. For a subset A of the group G we write $A^g = gAg^{-1}$ and $A^G = \{gag^{-1} \mid a \in A, g \in G\}$.

Remark 2.1 ([10, Remark 2.2]). Let $Q \subseteq G$ be quasiconvex and $g \in G$. Then the subsets gQ, Qg and gQg^{-1} are quasiconvex.

Thus each conjugate of a quasiconvex subgroup in a hyperbolic group is again quasiconvex. Another important property of hyperbolic groups states that any cyclic subgroup is quasiconvex (see e.g. [1]).

We will also use the concept of *Gromov boundary* of a hyperbolic group G; for a detailed account the reader is referred to the relevant chapters in [4] or [1]. A sequence $(x_i)_{i \in \mathbb{N}}$ of elements of the group G is said to be *convergent to infinity* if

$$\lim_{i, j \to \infty} (x_i | x_j)_{1_G} = \infty.$$

Two sequences $(x_i)_{i \in \mathbb{N}}$, $(y_j)_{i \in \mathbb{N}}$ convergent to infinity are said to be equivalent if

$$\lim_{i \to \infty} (x_i | y_i)_{1_G} = \infty.$$

The points of the boundary ∂G are identified with the equivalence classes of sequences convergent to infinity. It is easy to see that this definition does not depend on the choice of a base-point: instead of 1_G one could use any fixed point p of $\Gamma(G, \mathscr{A})$; see [1]. If α is the equivalence class of $(x_i)_{i \in \mathbb{N}}$, we will write $\lim_{i \to \infty} x_i = \alpha$.

The space ∂G can be topologized so that it becomes compact, Hausdorff and metrizable (see [4], [1]). Left multiplication by elements of G induces a homeomorphic action of G on its boundary: for any $g \in G$ and $[(x_i)_{i \in \mathbb{N}}] \in \partial G$ set

$$g \circ [(x_i)_{i \in \mathbb{N}}] := [(gx_i)_{i \in \mathbb{N}}] \in \partial G.$$

If $g \in G$ has infinite order then the sequences $(g^i)_{i \in \mathbb{N}}$ and $(g^{-i})_{i \in \mathbb{N}}$ converge to infinity and we will use the notation

$$\lim_{i \to \infty} g^i = g^{\infty} \in \partial G, \quad \lim_{i \to \infty} g^{-i} = g^{-\infty} \in \partial G.$$

The *limit set* $\Lambda(A)$ of a subset $A \subseteq G$ is the collection of points $\alpha \in \partial G$ that are limits of sequences (convergent to infinity) from A.

We require an auxiliary binary relation between subsets of an arbitrary group G defined in [14]. For $A, B \subseteq G$, write $A \preceq B$ if and only if there exist elements $x_1, \dots, x_n \in G$ such that

$$A \subseteq Bx_1 \cup Bx_2 \cup \dots \cup Bx_n.$$

It is not difficult to see that the relation $\preceq$ is transitive and that for any $g \in G$, $A \preceq B$ implies $gA \preceq gB$.

Lemma 2.3 ([14, Lemma 2.1]). *Let A, B be subgroups of G. Then $A \preceq B$ if and only if the index $|A : (A \cap B)|$ is finite.*

The basic properties of limit sets are described in the following statement:

Lemma 2.4 ([7], [22], [14, Lemma 6.2]). *Suppose that A, B are arbitrary subsets of G and $g \in G$. Then*

(a) $\Lambda(A) = \varnothing$ *if and only if A is finite*;

(b) $\Lambda(A)$ *is a closed subset of the boundary* ∂G;

(c) $\Lambda(A \cup B) = \Lambda(A) \cup \Lambda(B)$;

(d) $\Lambda(Ag) = \Lambda(A)$, $g \circ \Lambda(A) = \Lambda(gA)$;

(e) $A \preceq B$ *implies* $\Lambda(A) \subseteq \Lambda(B)$.

The following property of limit sets of quasiconvex subgroups was first proved by Swenson:

Lemma 2.5 ([22, Theorem 8], [11, Lemma 9.1]). *Let A, B be quasiconvex subgroups of a hyperbolic group G. Then $\Lambda(A) \cap \Lambda(B) = \Lambda(A \cap B)$ in ∂G.*

3 Auxiliary facts

Lemma 3.1. *Assume that H is an η-quasiconvex subgroup of a δ-hyperbolic group G, X is a word over $\mathcal{A}$ representing an element of infinite order in G, $0 < \lambda \leqslant 1$ and $c \geqslant 0$. Let $v = v(\delta, \lambda, c)$ be the constant given by Lemma 2.1. There exists $N = N(\delta, \eta, v, G) \in \mathbb{N}$ such that for any $m \in \mathbb{N}$ the following property holds.*

If a word $W \equiv U_1 X^n U_2$ is (λ, c)-quasi-geodesic and satisfies

$$\|U_1\|, \|U_2\| > (m + \nu + c)/\lambda, \quad n \geqslant N \quad \text{and} \quad \mathrm{elem}(V_1 W V_2) \in H$$

for words V_1, V_2 with $\|V_1\|, \|V_2\| \leqslant m$, then there exist $k \in \mathbb{N}$ and $a \in G$ such that $\mathrm{elem}(X^k) \in H^a$ *and* $|a|_G \leqslant 2\delta + \nu + \eta$.

Proof. Consider a path q in $\Gamma(G, \mathscr{A})$ starting at 1_G and labelled by $V_1 W V_2$. By our assumptions, $q_+ \in H$. Let p and r be its (λ, c)-quasi-geodesic subpaths with $\mathrm{lab}(p) \equiv W$ and $\mathrm{lab}(r) \equiv X^n$ respectively. Choose an arbitrary *phase* vertex $u \in r$ such that the subpath of r from r_- to u is labelled by some power of X.

By Lemma 2.1 we can find $v \in [p_-, p_+]$ satisfying $d(u, v) \leqslant \nu$. Our assumptions and the triangle inequality give

$$d(p_-, v) \geqslant d(p_-, u) - d(u, v) \geqslant \lambda\|U_1\| - c - \nu > m,$$
$$d(p_+, v) \geqslant d(p_+, u) - d(u, v) \geqslant \lambda\|U_2\| - c - \nu > m.$$

Hence $v \in \mathcal{O}_{2\delta}([1_G, q_+])$ by Lemma 2.2. Thus

$$u \in \mathcal{O}_{2\delta+\nu}([1_G, q_+]) \subset \mathcal{O}_{2\delta+\nu+\eta}(H),$$

i.e., there is an element $a = a(u) \in G$ such that $|a|_G \leqslant 2\delta + \nu + \eta$ and $u \in Ha$.

Now, since the alphabet $\mathscr{A}$ is finite, there are only finitely many elements in G having length at most $2\delta + \nu + \eta$. Hence, if n is large enough, there will be two different phase vertices $u_1, u_2 \in r$ with $a(u_1) = a(u_2) = a$. By the construction,

$$u_1^{-1} u_2 = \mathrm{elem}(X^k) \in a^{-1} H a$$

for some positive integer k (X^k is a label of the segment of r from u_1 to u_2, provided that these points are chosen in the correct order). $\square$

Lemma 3.2. *Assume that G is a hyperbolic group and $H \leqslant G$ is a quasiconvex subgroup. If $g \in G$ and $gH \preceq H$, then $H \preceq gH$.*

Proof. If H is finite, the statement is trivial. Our assumptions imply that

$$g^{k-1} H \preceq g^{k-2} H \preceq \cdots \preceq gH \preceq H$$

for all $k \in \mathbb{N}$. Hence

$$g^{k-1} H \preceq H. \tag{1}$$

If $g \in G$ has finite order k, then we obtain the desired result by multiplying both sides of the above formula by g.

Thus we can further assume that H is infinite and that g has infinite order. Therefore H has at least one limit point $\alpha \in \Lambda(H)$. Observe that (1) implies

$$g^n \circ \Lambda(H) = \Lambda(g^n H) \subseteq \Lambda(H),$$

and thus $g^n \circ \alpha \in \Lambda(H)$ for all $n \in \mathbb{N}$.

It is well known that if $\alpha \neq g^{-\infty}$ in ∂G then the sequence $(g^n \circ \alpha)_{n \in \mathbb{N}}$ converges to g^{∞} (see, for instance, [4, (8.16)]). Since $\Lambda(H)$ is a closed subset of ∂G, in each case we have

$$\Lambda(\langle g \rangle_{\infty}) = \{g^{\infty}, g^{-\infty}\} \cap \Lambda(H) \neq \emptyset.$$

By Lemma 2.5, this implies that $g^k \in H$ for some $k \in \mathbb{N}$. Combining this fact with (1) we get $H = g^k H \preceq gH$, which concludes the proof. $\square$

The previous lemma has the following consequence:

Lemma 3.3. *Suppose that H, K are subgroups of a hyperbolic group G such that $H \leqslant K$, $|K : H| = \infty$ and H is quasiconvex. Then $|K : (K \cap H^g)| = \infty$ for any $g \in G$.*

Proof. If $|K : (K \cap H^g)| < \infty$ for some $g \in G$ then $H \preceq K \preceq gHg^{-1} \preceq gH$ by Lemma 2.3. Consequently $g^{-1}H \preceq H$. Hence $H \preceq g^{-1}H$ by Lemma 3.2, implying that $gHg^{-1} \preceq gH \preceq H$. But the latter leads to $K \preceq H$, which contradicts the condition $|K : H| = \infty$ (see Lemma 2.3). $\square$

If G is a hyperbolic group, then each element $g \in G$ of infinite order belongs to a unique *maximal elementary subgroup* $E(g)$. By [17, Lemmas 1.16, 1.17] this subgroup has the following description:

$$\begin{aligned} E(g) &= \{x \in G \mid xg^k x^{-1} = g^l \text{ for some } k, l \in \mathbb{Z} \backslash \{0\}\} \\ &= \{x \in G \mid xg^n x^{-1} = g^{\pm n} \text{ for some } n \in \mathbb{N}\}. \end{aligned} \tag{2}$$

Note that the subgroup $E^+(g) := \{x \in G \mid xg^n x^{-1} = g^n \text{ for some } n \in \mathbb{N}\}$ has index at most 2 in $E(g)$.

Let $W_1, W_2, \dots, W_l$ be words in $\mathscr{A}$ representing elements $g_1, g_2, \dots, g_l$ of infinite order, where $E(g_i) \neq E(g_j)$ for $i \neq j$. The following lemma will be useful:

Lemma 3.4 ([17, Lemma 2.3]). *There exist constants $\lambda = \lambda(W_1, W_2, \dots, W_l) > 0$, $c = c(W_1, W_2, \dots, W_l) \geqslant 0$ and $N = N(W_1, W_2, \dots, W_l) > 0$ such that any path p in the Cayley graph $\Gamma(G, \mathscr{A})$ with label $W_{i_1}^{m_1} W_{i_2}^{m_2} \dots W_{i_s}^{m_s}$ is (λ, c)-quasi-geodesic if $i_k \neq i_{k+1}$ for $k = 1, 2, \dots, s-1$, and $|m_k| > N$ for $k = 2, 3, \dots, s-1$.*

For a subgroup H of G denote by H^0 the set of elements of infinite order in H; if $A \subseteq G$, we write $C_H(A)$ for the centralizer of A in H.

Set $E(H) = \bigcap_{x \in H^0} E(x)$. If H is a non-elementary subgroup of G, then $E(H)$ is the unique maximal finite subgroup of G normalized by H (see [17, Proposition 1]). If $g \in G^0$, we write $T(g)$ for the set of elements of finite order in $E(g)$.

Let G be a hyperbolic group and H a non-elementary subgroup. Recalling the definition from [17] (and using terminology from [12]), we say that an element $g \in H^0$ is *H-suitable* if

$$E(H) = T(g) \quad \text{and} \quad E(g) = E^+(g) = C_G(g) = T(g) \times \langle g \rangle.$$

In particular, if g is H-suitable then $g \in C_H(E(H))$.

Two elements $g, h \in G$ of infinite order are called *commensurable* if $g^k = ah^l a^{-1}$ for non-zero integers k, l and some $a \in G$. The following important statement was proved by Ol'shanskii in 1993:

Lemma 3.5 ([17, Lemma 3.8]). *Every non-elementary subgroup H of a hyperbolic group G contains an infinite set of pairwise non-commensurable H-suitable elements.*

Suitable elements can be modified in a natural way:

Lemma 3.6 ([12, Lemma 4.3]). *Let H be a non-elementary subgroup of a hyperbolic group G, and g be an H-suitable element. If $y \in C_H(E(H)) \backslash E(g)$ then there exists $N \in \mathbb{N}$ such that the element yg^n has infinite order in H and is H-suitable for every $n \geqslant N$.*

In [14], the author studied properties of quasiconvex subgroups of infinite index:

Lemma 3.7 ([14, Proposition 1]). *Suppose that H is a quasiconvex subgroup of a hyperbolic group G and that K is a subgroup of G that satisfies $|K : (K \cap H^g)| = \infty$ for all $g \in G$. Then K has an element x of infinite order such that $\langle x \rangle_\infty \cap H^G = \{1_G\}$.*

Later we will utilize a stronger result:

Lemma 3.8. *Assume that H, K are subgroups of a δ-hyperbolic group G such that H is η-quasiconvex, K is non-elementary and $|K : (K \cap H^g)| = \infty$ for every $g \in G$. Then K has a K-suitable element y such that $\langle y \rangle_\infty \cap H^G = \{1_G\}$.*

Proof. Set $K' = C_K(E(K))$. Since $E(K)$ is a finite subgroup normalized by K, we have $|K : K'| < \infty$. Therefore $|K' : (K' \cap H^g)| = \infty$ for all $g \in G$. Applying Lemma 3.7, we can find an element of infinite order $x \in K'$ such that $\langle x \rangle_\infty \cap H^G = \{1_G\}$. By Lemma 3.5 there is a K-suitable element $z \in K$ which is non-commensurable with x, and hence $\langle x \rangle_\infty \cap E(z) = \{1_G\}$.

Choose words X, Z in the alphabet $\mathscr{A}$ representing x, z. Then we can find $\lambda = \lambda(X, Z)$, $c = c(X, Z)$ and $N_1 = N_1(X, Z)$ as in Lemma 3.4. Define $v = v(\delta, \lambda, c)$ and $N_2 = N_2(\delta, \eta, v, G)$ as in Lemmas 2.1 and 3.1. Write $N = \max\{N_1, N_2\}$ and apply Lemma 3.6 to obtain $n \geqslant N_1$ such that the element $y = x^N z^n \in K$ is K-suitable.

It remains to check that $\langle y\rangle_\infty \cap H^G = \{1_G\}$. Assume otherwise, i.e., there exist $t \in \mathbb{N}$ and $g \in G$ such that $y^t \in H^g$. Then for each $l \in \mathbb{N}$, the element $y^{tl} \in H^g$ will be represented by the (λ, c)-quasi-geodesic word $W \equiv (X^N Z^n)^{tl}$. If l is chosen sufficiently large (compared with $m = |g^{-1}|_G = |g|_G$), then we can find a subword of the form X^N in the 'middle' of W which satisfies all assumptions of Lemma 3.1. Hence $x^k = \mathrm{elem}(X^k) \in H^G$ for some $k \in \mathbb{N}$. This contradicts the construction of x. Thus the lemma is proved. $\square$

As usual, let G be a δ-hyperbolic group and H an η-quasiconvex subgroup.

Lemma 3.9. *Suppose that the elements $x_1, x_2 \in G$ have infinite order, that $E(x_1) \neq E(x_2)$ and that $\langle x_i\rangle_\infty \cap H^G = \{1_G\}$ for $i = 1, 2$. Then there exists $N \in \mathbb{N}$ such that for any $m, n \geqslant N$ the elements x_1^m, x_2^n freely generate a free subgroup F of rank 2 in G. Moreover, F is quasiconvex and $F \cap H^G = \{1_G\}$.*

Proof. Choose words X_1, X_2 in the alphabet $\mathscr{A}$ representing x_1, x_2. Apply Lemma 3.4 to find the corresponding $\lambda = \lambda(X_1, X_2)$, $c = c(X_1, X_2)$ and $N_1 = N_1(X_1, X_2)$. Then one can find the constant $v = v(\delta, \lambda, c)$ from Lemma 2.1 and define $N_2 = N_2(\delta, \eta, v, G)$ in accordance with Lemma 3.1.

Set $N = \max\{N_1, N_2, \lfloor c/\lambda \rfloor + 1\}$, and consider arbitrary integers $m, n \geqslant N$ and the subgroup $F = \langle x_1^m, x_2^n\rangle \leqslant G$. By Lemma 3.4 any non-empty (freely) reduced word W in the generators $\{X_1^m, X_2^n\}$ is (λ, c)-quasi-geodesic. Hence

$$|\mathrm{elem}(W)|_G \geqslant \lambda \|W\| - c \geqslant \lambda N - c > 0.$$

Consequently, $\mathrm{elem}(W) \neq 1_G$ in G, implying that F is free with free generating set $\{x_1^m, x_2^n\}$. By the construction of v, F will be ε-quasiconvex, where

$$\varepsilon = v + \frac{1}{2}\max\{m\|X_1\|, n\|X_2\|\}.$$

Consider a non-empty cyclically reduced word W in the generators $\{X_1^m, X_2^n\}$. To establish the last claim, it is sufficient to demonstrate that $\mathrm{elem}(W) \notin H^G$. Arguing as in Lemma 3.8, suppose that $\mathrm{elem}(W) \in H^g$ for some $g \in G$. Then $(\mathrm{elem}(W))^l \in H^g$ for every $l \in \mathbb{N}$. Choosing l sufficiently large and applying Lemma 3.1 one obtains a contradiction to the assumption that $\langle x_i\rangle_\infty \cap H^G = \{1_G\}$ for $i = 1, 2$, as in Lemma 3.8. Therefore $F \cap H^G = \{1_G\}$. $\square$

Corollary 2. *With the assumptions of Lemma 3.8, K has a free subgroup F of rank 2 which is quasiconvex in G, such that $E(F) = E(K)$ and $F \cap H^G = \{1_G\}$.*

Proof. Choose a K-suitable element $x_1 \in K$ as given by Lemma 3.8. Since K is non-elementary, there exists $y \in K \backslash E(x_1)$. Therefore $x_2 := y x_1 y^{-1} \in K^0$ and $E(x_2) \neq E(x_1)$ (cf. (2)). By the construction, we have $\langle x_i\rangle_\infty \cap H^G = \{1_G\}$ for $i = 1, 2$. Hence F can be found by applying Lemma 3.9. Evidently $E(K) \leqslant E(F)$, and $E(F) \subseteq T(x_1) = E(K)$. Thus $E(F) = E(K)$. $\square$

4 Free products of quasiconvex subgroups

Below we assume that G is a δ-hyperbolic group generated by a finite set $\mathcal{A}$. First let us recall some properties of the hyperbolic boundary.

Lemma 4.1 ([12, Lemma 2.14]). *Suppose that A, B are arbitrary subsets of G and $\Lambda(A) \cap \Lambda(B) = \varnothing$. Then* $\sup_{a \in A, b \in B}\{(a|b)_{1_G}\} < \infty$.

Remark 4.1. Suppose that $g, x \in G$ and g has infinite order. If in ∂G we have $(x \circ \{g^{\pm\infty}\}) \cap \{g^{\pm\infty}\} \neq \varnothing$, then $x \in E(g)$. If $E(g) = E^+(g)$ then $g^\infty \notin G \circ \{g^{-\infty}\}$.

Note that $x \circ \{g^{\pm\infty}\} = \{(xgx^{-1})^{\pm\infty}\} = \Lambda(\langle xgx^{-1}\rangle) \subset \partial G$. Since any cyclic subgroup in a hyperbolic group is quasiconvex, we can apply Lemma 2.5 to show that $\langle g\rangle \cap \langle xgx^{-1}\rangle \neq \{1_G\}$. Hence $x \in E(g)$.

Let $E(g) = E^+(g)$ and suppose that $g^\infty = x \circ g^{-\infty}$ for some $x \in G$. Then, as shown above, $x \in E(g) = E^+(g)$. Hence

$$x \circ g^{-\infty} = \lim_{n \to -\infty} (xgx^{-1})^n = \lim_{n \to -\infty} g^n = g^{-\infty}.$$

Thus we obtain a contradiction to the inequality $g^\infty \neq g^{-\infty}$.

Lemma 4.2. *Let $g, x \in G$, where g has infinite order and $E(g) = E^+(g)$. Then there is $N_1 \in \mathbb{N}$ such that for every $n \geqslant N_1$ the element $xg^n \in G$ has infinite order.*

Proof. If $x \notin E(g)$ then the claim follows by [11, Lemma 9.14].

So assume that $x \in E(g)$. Since $E(g) = E^+(g)$, the center of $E(g)$ has finite index in $E(g)$, and thus $E(g)$ is an FC-group. By a theorem of B. H. Neumann [15] the elements of finite order form a subgroup $T(g)$ of $E(g)$. Therefore the cardinality of the intersection $\{xg^k \mid k \in \mathbb{Z}\} \cap T(g)$ can be at most 1. Thus $xg^n \notin T(g)$ for each sufficiently large n. $\square$

The main result of this paper is based on the following statement concerning broken lines in a δ-hyperbolic metric space:

Lemma 4.3 ([18, Lemma 21], [14, Lemma 3.5]). *Let $p = [y_0, y_1, \dots, y_n]$ be a broken line in $\Gamma(G, \mathcal{A})$ such that $\|[y_{i-1}, y_i]\| > C_1$ for $i = 1, \dots, n$, and $(y_{i-1}|y_{i+1})_{y_i} < C_0$ for $i = 1, \dots, n-1$, where $C_0 \geqslant 14\delta$ and $C_1 > 12(C_0 + \delta)$. Then the geodesic segment $[y_0, y_n]$ is contained in the closed 14δ-neighborhood of p and $\|[y_0, y_n]\| \geqslant \|p\|/2$.*

Suppose that a, b, c, d are arbitrary points in $\Gamma(G, \mathcal{A})$. Considering the geodesic triangle with the vertices 1_G, a, ab, one observes that

$$(a \mid ab)_{1_G} = |a|_G - (1_G \mid ab)_a = |a|_G - (a^{-1}|b)_{1_G}.$$

Now, since $\Gamma(G, \mathscr{A})$ is δ-hyperbolic,

$$(a|c)_{1_G} \geqslant \min\{(a \mid ab)_{1_G}, (ab \mid c)_{1_G}\} - \delta = \min\{|a|_G - (a^{-1}|b)_{1_G}, (ab \mid c)_{1_G}\} - \delta. \quad (3)$$

Replacing c by cd in the above formula, we get

$$(a \mid cd)_{1_G} \geqslant \min\{|a|_G - (a^{-1}|b)_{1_G}, (ab \mid cd)_{1_G}\} - \delta. \quad (4)$$

Since the Gromov product is symmetric, we can combine (3) and (4) to obtain

$$\begin{aligned}(a|c)_{1_G} &\geqslant \min\{|c|_G - (c^{-1}|d)_{1_G}, (a \mid cd)_{1_G}\} - \delta \\ &\geqslant \min\{|a|_G - (a^{-1}|b)_{1_G}, |c|_G - (c^{-1}|d)_{1_G}, (ab \mid cd)_{1_G}\} - 2\delta.\end{aligned} \quad (5)$$

Theorem 5. *Consider elements $g_1, x_1, g_2, x_2, \ldots, g_s, x_s \in G$ and an η-quasiconvex subgroup H. Suppose that the following three conditions are satisfied*:

(i) *$g_1, \ldots, g_s$ have infinite order and are pairwise non-commensurable*;

(ii) *$E(g_i) = E^+(g_i)$ for $i = 1, 2, \ldots, s$*;

(iii) *$E(g_i) \cap H = E(g_i) \cap x_i^{-1} H x_i = \{1_G\}$ for $i = 1, 2, \ldots, s$.*

Then there exists $N \in \mathbb{N}$ such that for every $n \geqslant N$ the elements $x_1 g_1^n, \ldots, x_s g_s^n$ of G have infinite order, and the subgroup

$$M := \langle H, x_1 g_1^n, \ldots, x_s g_s^n \rangle \leqslant G$$

is quasiconvex in G and isomorphic (*in the canonical way*) *to the free product $H * \langle x_1 g_1^n \rangle * \cdots * \langle x_s g_s^n \rangle$.*

Proof. Choose arbitrary elements $w_1, w_2 \in M$ and define $w = w_1^{-1} w_2 \in M$. Then we can write

$$w = h_0 (x_{i_1} g_{i_1}^n)^{\epsilon_1} h_1 (x_{i_2} g_{i_2}^n)^{\epsilon_2} \ldots h_{l-1} (x_{i_l} g_{i_l}^n)^{\epsilon_l} h_l, \quad (6)$$

where $h_j \in H$, $i_j \in \{1, \ldots, s\}$, $\epsilon_j \in \{1, -1\}$, $j = 1, 2, \ldots, l$, $l \in \mathbb{N} \cup \{0\}$. We can assume that the representation (6) is reduced in the following sense: if $1 \leqslant j \leqslant l-1$, $i_j = i_{j+1}$ and $\epsilon_{j+1} = -\epsilon_j$ then $h_j \neq 1_G$.

Consider a geodesic broken line $[y_0, y_1, \ldots, y_{l+1}]$ in $\Gamma(G, \mathscr{A})$ with $y_0 = w_1$ and $\operatorname{elem}([y_k, y_{k+1}]) = h_k (x_{i_{k+1}} g_{i_{k+1}}^n)^{\epsilon_{k+1}}$ for $k = 0, 1, \ldots, l-1$ and $\operatorname{elem}([y_l, y_{l+1}]) = h_l$. Thus $\operatorname{elem}([y_0, y_{l+1}]) = w$ and $y_{l+1} = w_2$.

Now we find upper bounds for the Gromov products

$$(y_{k-1} | y_{k+1})_{y_k} = (y_k^{-1} y_{k-1} \mid y_k^{-1} y_{k+1})_{1_G}, \quad \text{for } k = 1, \ldots, l.$$

By the assumptions of the theorem, Lemma 2.5 implies that

$$x_i \circ g_i^{\infty} = (x_i g_i x_i^{-1})^{\infty} \notin \Lambda(H) \quad \text{and} \quad g_i^{-\infty} \notin \Lambda(H), \quad \text{for } i = 1, \ldots, s.$$

Since

$$\Lambda(\{x_i g_i^m, g_i^{-m} x_i^{-1} \mid m \in \mathbb{N}, 1 \leqslant i \leqslant s\}) = \{x_i \circ g_i^{\infty}, g_i^{-\infty} \mid 1 \leqslant i \leqslant s\} \subset \partial G,$$

we can apply Lemma 4.1 and define

$$\alpha := \max\{(h \mid x_i g_i^m)_{1_G}, (h \mid g_i^{-m} x_i^{-1})_{1_G} \mid h \in H, 1 \leqslant i \leqslant s, m \in \mathbb{N}\} < \infty.$$

Similarly, since g_i and g_j are non-commensurable if $i \neq j$ and $E(g_i) = E^+(g_i)$, we have (from Lemma 2.5 and Remark 4.1)

$$G \circ \{g_i^{\pm\infty}\} \cap G \circ \{g_j^{\pm\infty}\} = \varnothing, \quad G \circ \{g_i^{\infty}\} \cap G \circ \{g_i^{-\infty}\} = \varnothing,$$

for all distinct $i, j \in \{1, \dots, s\}$. Hence the following numbers are also finite:

$$\begin{aligned}\beta_1 := \max\{&((x_i g_i^m)^{-1} \mid h x_j g_j^t)_{1_G}, (x_i g_i^m \mid h(x_j g_j^t)^{-1})_{1_G} \mid h \in H,\\ &|h|_G \leqslant 2\alpha + 2\delta, 1 \leqslant i, j \leqslant s, m, t \in \mathbb{N}\},\\ \beta_2 := \max\{&(x_i g_i^m \mid h x_j g_j^t)_{1_G}, ((x_i g_i^m)^{-1} \mid h(x_j g_j^t)^{-1})_{1_G} \mid h \in H,\\ &|h|_G \leqslant 2\alpha + 2\delta, 1 \leqslant i, j \leqslant s, i \neq j, m, t \in \mathbb{N}\}.\end{aligned}$$

Note that if $h \in H \backslash \{1_G\}$ then our assumptions imply that $x_i^{-1} h x_i \notin E(g_i)$. Therefore $\{g_i^{\pm\infty}\} \cap (x_i^{-1} h x_i) \circ \{g_i^{\pm\infty}\} = \varnothing$ from Remark 4.1, implying that $x_i \circ g_i^{\infty} \neq (hx_i) \circ g_i^{\infty}$ for $i = 1, \dots, s$. Remark 4.1 also shows that $g_i^{-\infty} \neq h \circ g_i^{-\infty}$ for each i. Consequently, by Lemma 4.1,

$$\begin{aligned}\beta_3 := \max\{&(x_i g_i^m \mid h x_i g_i^t)_{1_G}, ((x_i g_i^m)^{-1} \mid h(x_i g_i^t)^{-1})_{1_G} \mid h \in H,\\ &|h|_G \leqslant 2\alpha + 2\delta, h \neq 1_G, 1 \leqslant i \leqslant s, m, t \in \mathbb{N}\} < \infty.\end{aligned}$$

Finally, define $\beta = \max\{\beta_1, \beta_2, \beta_3\}$,

$$C_0 = \max\{\alpha + 2\delta, \beta + \delta, 14\delta\} + 1 \quad \text{and} \quad C_1 = 12(C_0 + \delta) + 1. \tag{7}$$

Since $g_1, \dots, g_s$ have infinite order there exists $N \in \mathbb{N}$ such that for each $i \in \{1, \dots, s\}$ one has

$$|g_i^n|_G > \max\{\alpha, \beta, 2C_1\} + \alpha + |x_i|_G + 2\delta, \quad \text{for all } n \geqslant N. \tag{8}$$

By Lemma 4.2 we can also assume that $x_1 g_1^n \dots x_s g_s^n$ have infinite order for all $n \geqslant N$.

Fix an integer $n \geqslant N$ and choose any $k \in \{1, \dots, l-1\}$. Then

$$(y_{k-1} | y_{k+1})_{y_k} = ((x_{i_k} g_{i_k}^n)^{-\epsilon_k} h_{k-1}^{-1} \mid h_k (x_{i_{k+1}} g_{i_{k+1}}^n)^{\epsilon_{k+1}})_{1_G}.$$

To simplify the notation, we write

$$a = (x_{i_k} g_{i_k}^n)^{-\epsilon_k}, \quad b = h_{k-1}^{-1}, \quad c = h_k \quad \text{and} \quad d = (x_{i_{k+1}} g_{i_{k+1}}^n)^{\epsilon_{k+1}}.$$

By construction,

$$(a|c)_{1_G}, (a^{-1}|b)_{1_G}, (c^{-1}|d)_{1_G} \leqslant \alpha. \tag{9}$$

We need to consider two separate cases.

Case 1. $|h_k|_G = |c|_G \leqslant 2\alpha + 2\delta$. Then, from the definition of β, we have $(a \mid cd)_{1_G} \leqslant \beta$. Therefore, applying formulas (4) and (9), one obtains

$$\begin{aligned} \beta &\geqslant \min\{|a|_G - (a^{-1}|b)_{1_G}, (ab \mid cd)_{1_G}\} - \delta \\ &\geqslant \min\{|g_{i_k}^n|_G - |x_{i_k}|_G - \alpha, (ab \mid cd)_{1_G}\} - \delta. \end{aligned}$$

By (8) we have $|g_{i_k}^n|_G - |x_{i_k}|_G - \alpha > \beta + \delta$, and hence the above inequality gives

$$(y_{k-1}|y_{k+1})_{y_k} = (ab \mid cd)_{1_G} \leqslant \beta + \delta < C_0.$$

Case 2. $|h_k|_G = |c|_G > 2\alpha + 2\delta$. Applying formulas (6) and (9) we obtain

$$\alpha \geqslant \min\{|a|_G - \alpha, |c|_G - \alpha, (ab \mid cd)_{1_G}\} - 2\delta.$$

Observing that

$$|a|_G - \alpha \geqslant |g_{i_k}^n|_G - |x_{i_k}|_G - \alpha > \alpha + 2\delta \quad \text{and} \quad |c|_G - \alpha > \alpha + 2\delta,$$

we conclude that

$$(y_{k-1}|y_{k+1})_{y_k} = (ab \mid cd)_{1_G} \leqslant \alpha + 2\delta < C_0.$$

Finally, we estimate the product $(y_{l-1}|y_{l+1})_{y_l} = ((x_{i_l} g_{i_l}^n)^{-\epsilon_k} h_{l-1}^{-1} \mid h_l)_{1_G}$. Write $a = (x_{i_l} g_{i_l}^n)^{-\epsilon_k}$, $b = h_{l-1}^{-1}$ and $c = h_l$.

Using formula (3) and the definition of α one obtains

$$\alpha \geqslant (a|c)_{1_G} \geqslant \min\{|x_{i_l} g_{i_l}^n|_G - \alpha, (ab \mid c)_{1_G}\} - \delta.$$

As before, this inequality implies that

$$(y_{l-1}|y_{l+1})_{y_l} = (ab \mid c)_{1_G} \leqslant \alpha + \delta < C_0.$$

Thus we have shown that

$$(y_{k-1}|y_{k+1})_{y_k} < C_0 \quad \text{for } k = 1, 2, \ldots, l. \tag{10}$$

Now we need to find a lower bound for the lengths of the sides in the broken line $[y_0, y_1, \dots, y_{l+1}]$.

Let $0 \leqslant k \leqslant l-1$. Note that $|ab|_G \geqslant |b|_G - (a^{-1}|b)_{1_G}$ for any $a, b \in G$. Hence

$$\begin{aligned}\|[y_k, y_{k+1}]\| &= |h_k(x_{i_{k+1}} g^n_{i_{k+1}})^{\epsilon_{k+1}}|_G \\ &\geqslant |(x_{i_{k+1}} g^n_{i_{k+1}})^{\epsilon_{k+1}}|_G - (h_k^{-1} \,|\, (x_{i_{k+1}} g^n_{i_{k+1}})^{\epsilon_{k+1}})_{1_G} \\ &\geqslant |g^n_{i_{k+1}}|_G - |x_{i_{k+1}}|_G - \alpha.\end{aligned}$$

Applying inequality (8) we obtain

$$\|[y_k, y_{k+1}]\| > 2C_1 \quad \text{if } 0 \leqslant k \leqslant l-1. \tag{11}$$

Two cases arise, depending on the value of $\|[y_l, y_{l+1}]\| = |h_l|_G$.

Case 1. $\|[y_l, y_{l+1}]\| = |h_l|_G \leqslant C_1$. Then we can use inequalities (10) and (11) to apply Lemma 4.3 to the geodesic broken line $p' = [y_0, \dots, y_l]$. Hence $[y_0, y_l] \subset \mathcal{O}_{14\delta}(p')$ and $d(y_0, y_l) \geqslant \|p'\|/2$.

Since geodesic triangles in $\Gamma(G, \mathcal{A})$ are δ-slim, we have

$$[y_0, y_{l+1}] \subset \mathcal{O}_\delta([y_0, y_l] \cup [y_l, y_{l+1}]) \subset \mathcal{O}_{\delta+C_1}([y_0, y_l]) \subset \mathcal{O}_{15\delta+C_1}(p').$$

Now if $l \geqslant 1$ in the presentation (6), one can use (11) to obtain

$$d(y_0, y_{l+1}) \geqslant d(y_0, y_l) - d(y_l, y_{l+1}) \geqslant \|p'\|/2 - C_1 \geqslant \|[y_0, y_1]\|/2 - C_1 > 0.$$

Case 2. $\|[y_l, y_{l+1}]\| = |h_l|_G > C_1$. Then we can apply Lemma 4.3 to the broken line $p = [y_0, \dots, y_l, y_{l+1}]$, thus obtaining

$$[y_0, y_{l+1}] \subset \mathcal{O}_{14\delta}(p).$$

As before, if $l \geqslant 1$ one has

$$d(y_0, y_{l+1}) \geqslant \|p\|/2 > 0.$$

Thus in each case we have established the following properties:

$$[y_0, y_{l+1}] \subset \mathcal{O}_{15\delta+C_1}(p), \tag{12}$$

$$d(y_0, y_{l+1}) > 0. \tag{13}$$

The inequality (13) implies that $w \neq 1_G$ in the group G for any element $w \in M$ having a 'reduced' presentation (6) with $l \geqslant 1$. Therefore

$$M \cong H * \langle x_1 g_1^n \rangle * \dots * \langle x_s g_s^n \rangle.$$

As $n \geqslant N$ is fixed, one can define the constants

$$\zeta = \max_{1 \leqslant i \leqslant s} \{|x_i g_i^n|_G\} < \infty \quad \text{and} \quad \varepsilon = 16\delta + C_1 + \eta + \zeta.$$

We will finish the proof by showing that $[y_0, y_{l+1}] \subset \mathcal{O}_\varepsilon(M)$, which implies that M is ε-quasiconvex.

Let $k \in \{0, 1, \ldots, l\}$. Using the construction of $y_k \in M$ and ζ, and δ-hyperbolicity of the Cayley graph, we obtain

$$[y_k, y_{k+1}] \subset \mathcal{O}_{\delta+\zeta}([y_k, y_k h_k]). \tag{14}$$

Now H is η-quasiconvex, and therefore $[1_G, h_k] \subset \mathcal{O}_\eta(H)$. The metric in $\Gamma(G, \mathscr{A})$ is invariant under the action of G by left translations, and so

$$[y_k, y_k h_k] \subset \mathcal{O}_\eta(y_k H) \subset \mathcal{O}_\eta(M).$$

Combining this with (14) we conclude that

$$[y_k, y_{k+1}] \subset \mathcal{O}_{\delta+\eta+\zeta}(M) \quad \text{for each } k.$$

Finally, an application of (12) yields that

$$[y_0, y_{l+1}] \subset \mathcal{O}_{15\delta+C_1+\delta+\eta+\zeta}(M) = \mathcal{O}_\varepsilon(M),$$

as desired. $\square$

5 Hyperbolic groups with engulfing

Proof of Theorem 1. Since every elementary group is residually finite, it is sufficient to consider the case when G is non-elementary. Let $x_1, \ldots, x_s$ be the generators of G.

Define

$$K = \bigcap_{L \leqslant G, |G:L| < \infty} L;$$

then K is normal in G. Suppose that K is infinite. If K were elementary, then it would be quasiconvex (from Lemmas 3.4 and 2.1). Hence, from a result of Mihalik and Towle [9] (see also [14, Corollary 2]), K would have finite index in G, and so G would also be elementary. Therefore K cannot be elementary.

Now we can apply Lemma 3.5 to find pairwise non-commensurable K-suitable elements $g_1, \ldots, g_{s+1} \in K$. Since the trivial subgroup $H = \{1_G\} \leqslant G$ is quasiconvex, we can use Theorem 5 to show that the subgroups $M = \langle x_1 g_1^n, \ldots, x_s g_s^n \rangle$ and $M' = \langle x_1 g_1^n, \ldots, x_s g_s^n, g_{s+1}^n \rangle$ are free (of ranks s and $(s+1)$ respectively) and quasiconvex in G for some sufficiently large $n \in \mathbb{N}$.

Note that M is a proper (infinite index) subgroup of G because $|M' : M| = \infty$. From our assumptions, G has a proper finite index subgroup L with $M \leqslant L$. By the construction, $K \leqslant L$, and thus $x_i g_i^n, g_i \in L$ for $i = 1, \dots, s$. Consequently $x_i \in L$ for $i = 1, \dots, s$, and this is a contradiction since $L \neq G$.

Therefore K is finite. If G is torsion-free then K is trivial, and thus G is residually finite. □

As $E(G)$ is the maximal finite normal subgroup in G, we obtain immediately the following statement:

Corollary 3. *With the assumptions of Theorem* 1, *suppose in addition that* $E(G) = \{1_G\}$. *Then* G *is residually finite.*

It will be convenient below to use the following equivalence relation between subsets of a group G defined in [14]: for any $A, B \subseteq G$ such that $A \preceq B$ and $B \preceq A$ we write $A \approx B$.

Remark 5.1 ([14, Remark 3]). If $A, B \subseteq G$, $A \approx B$ and A is quasiconvex, then B is also quasiconvex.

In particular, if $A \leqslant B$ are subgroups of G and A has finite index in B then $A \approx B$. Hence A is quasiconvex if and only if B is quasiconvex.

Lemma 5.1. *Let* G *be a residually finite hyperbolic group and let* $H \leqslant G$ *be a quasiconvex subgroup. Suppose that every proper quasiconvex subgroup of* G *is engulfed. Then* H *has finite index in its profinite closure* K *in* G.

Proof. Assume to the contrary that $|K : H| = \infty$. Since G is residually finite, any finite subset is closed in the profinite topology. Thus H is infinite; hence K is non-elementary.

Choose some generating set $x_1, \dots, x_s$ of G. Since G is residually finite, it has a finite index subgroup G_1 satisfying

$$G_1 \cap \left(E(K) \cup \bigcup_{i=1}^{s} x_i E(K) x_i^{-1} \right) = \{1_G\}. \tag{15}$$

Define $H_1 = H \cap G_1$; then $|H : H_1| < \infty$ and, according to Remark 5.1, H_1 is quasiconvex. The profinite closure K_1 of H_1 in G has finite index in K, and so K_1 is non-elementary and $|K_1 : H_1| = \infty$. The definition of a finite index subgroup implies that there is $l \in \mathbb{N}$ such that $g^l \in K_1$ for each $g \in K$. Since $E(g) = E(g^l)$ for any $g \in K^0$ we have

$$E(K) \subseteq E(K_1) = \bigcap_{g \in (K_1)^0} E(g) \subseteq \bigcap_{g \in K^0} E(g^l) = \bigcap_{g \in K^0} E(g) = E(K).$$

Thus $E(K_1) = E(K)$.

By Lemma 3.3, we have $|K_1 : (K_1 \cap H_1^g)| = \infty$ for every $g \in G$, and we can use Corollary 2 to obtain a free subgroup $F \leqslant K_1$ of rank 2 satisfying $F \cap H_1^G = \{1_G\}$ and $E(F) = E(K_1) = E(K)$.

By Lemma 3.5, there exist elements $g_1, \dots, g_{s+1} \in F$ which are pairwise non-commensurable and F-suitable. Consequently $E(g_i) = \langle g_i \rangle_\infty \times E(K)$ and, since $E(K)$ is finite, (15) implies that

$$E(g_i) \cap H_1 = E(K) \cap H_1 = \{1_G\}, \qquad \text{for } i = 1, 2, \dots, s+1,$$
$$E(g_i) \cap x_i^{-1} H_1 x_i = E(K) \cap x_i^{-1} H_1 x_i = \{1_G\}, \quad \text{for } i = 1, 2, \dots, s.$$

Now we apply Theorem 5 to find $n \in \mathbb{N}$ such that the subgroups

$$M = \langle H_1, x_1 g_1^n, \dots, x_s g_s^n \rangle \quad \text{and} \quad M' = \langle H_1, x_1 g_1^n, \dots, x_s g_s^n, g_{s+1}^n \rangle$$

are quasiconvex in G and $M' = M * \langle g_{s+1}^n \rangle_\infty \leqslant G$. Thus $|M' : M| = \infty$ and M is a proper subgroup of G.

Since M is engulfed by hypothesis, G has a proper finite index subgroup L containing M. We have $K_1 \leqslant L$ because $H_1 \leqslant M \leqslant L$, and, since $x_i g_i^n, g_i \in M \cup K_1 \subset L$, we conclude that $x_i \in L$ for $i = 1, \dots, s$, so that $G = L$, a contradiction. $\square$

We will now prove Theorem 3, which strengthens the statement of the previous lemma.

Proof of Theorem 3. We can assume that G is non-elementary because any elementary group is LERF. Since G is residually finite, there is a finite index subgroup $G_1 \leqslant G$ with $G_1 \cap E(G) = \{1_G\}$.

Take an arbitrary quasiconvex subgroup $H \leqslant G$ and set $H_1 = H \cap G_1$. From Remark 5.1, the subgroup H_1 is quasiconvex in G. Therefore, by Lemma 5.1, H_1 has finite index in its profinite closure K_1 in G.

If $H_1 = K_1$, i.e., if H_1 is closed in the profinite topology on G, then so is H. Thus we can suppose that $H_1 \neq K_1$. Consequently $|G : H_1| = \infty$, and thus $|G : K_1| = \infty$.

The subgroup K_1 is quasiconvex by Remark 5.1, and hence we can apply Lemma 3.8 to find a G-suitable element $g \in G$ such that $\langle g \rangle_\infty \cap K_1 = \{1_G\}$. Since $E(g) = \langle g \rangle \times E(G)$, $E(G)$ is finite and $K_1 \leqslant G_1$, we have

$$E(g) \cap K_1 = E(G) \cap K_1 = \{1_G\}.$$

Now we can apply Theorem 5 to find a number $n \in \mathbb{N}$ such that the subgroups $M = \langle H_1, g^n \rangle$ and $M' = \langle K_1, g^n \rangle$ are quasiconvex in G and $M' \cong K_1 * \langle g^n \rangle_\infty$.

Using properties of free products, we observe that $M \leqslant M'$ and $|M' : M| = \infty$ because $H_1 \lneqq K_1$. On the other hand, M' is contained in the profinite closure of M in G. Thus we obtain a contradiction to the assertion of Lemma 5.1. $\square$

Before proceeding with the next statement, we need to recall some facts concerning quasi-isometries of metric spaces. Let $\mathcal{X}$ and $\mathcal{Y}$ be metric spaces with metrics $d(\cdot,\cdot)$

and $e(\cdot,\cdot)$ respectively. A map $f : \mathcal{X} \to \mathcal{Y}$ is called a *quasi-isometry* if there are constants $D_1 > 0$ and $D_2 \geqslant 0$ such that

$$D_1^{-1} d(a,b) - D_2 \leqslant e(f(a), f(b)) \leqslant D_1 d(a,b) + D_2 \quad \text{for all } a, b \in \mathcal{X}.$$

The spaces $\mathcal{X}$ and $\mathcal{Y}$ are said to be *quasi-isometric* if there exists a quasi-isometry $f : \mathcal{X} \to \mathcal{Y}$ whose image is *quasi-dense* in $\mathcal{Y}$, i.e., there exists $\varepsilon \geqslant 0$ such that for each $y \in \mathcal{Y}$ there is $x \in \mathcal{X}$ with $e(y, f(x)) \leqslant \varepsilon$.

Gromov [5] showed that if $\mathcal{X}$ is hyperbolic and quasi-isometric to $\mathcal{Y}$ (through some map $f : \mathcal{X} \to \mathcal{Y}$) then $\mathcal{Y}$ is also hyperbolic. He also noted that in this case the image $f(Q)$ of any quasiconvex subset $Q \subseteq \mathcal{X}$ will be quasiconvex in $\mathcal{Y}$.

Proof of Theorem 2. Note that Q is a finite normal subgroup of G by Theorem 1.

Consider the quotient $G_1 = G/Q$ together with the natural homomorphism $\psi : G \to G_1$. Since Q is finite, ψ is a quasi-isometry between G and G_1 (when G_1 is equipped with the word metric induced by the image of the finite generating set of G). Therefore G_1 is also hyperbolic and any preimage map $\bar{\psi}^{-1} : G_1 \to G$ (which maps an element of G_1 to some element of G belonging to the corresponding left coset modulo Q) is a quasi-isometry as well.

Choose an arbitrary proper quasiconvex subgroup $H_1 \leqslant G_1$. Then $\bar{\psi}^{-1}(H_1)$ is a quasiconvex subset of G and

$$\bar{\psi}^{-1}(H_1) \subseteq \psi^{-1}(H_1) \subseteq \bar{\psi}^{-1}(H_1) \cdot Q,$$

where $\psi^{-1}(H_1)$ is the full preimage of H_1 in G.

As Q is finite, the above formula implies $\bar{\psi}^{-1}(H_1) \approx \psi^{-1}(H_1)$. Therefore $\psi^{-1}(H_1)$ is quasiconvex in G by Remark 5.1. By our assumptions, there is a proper finite index subgroup $L \leqslant G$ containing $\psi^{-1}(H_1)$. By definition, $Q \leqslant L$, and hence $\psi(L)$ is a proper finite index subgroup of G_1 with $H_1 \leqslant \psi(L)$.

Thus we have shown that G_1 also engulfs each proper quasiconvex subgroup. By the construction, G_1 is residually finite and therefore GFERF (from Theorem 3).

Consider any quasiconvex subgroup $H \leqslant G$. Then $\psi(H)$ is quasiconvex in G_1 and thus it is closed in the profinite topology on G_1. The homomorphism ψ is a continuous map if G and G_1 are equipped with their profinite topologies, and thus the full preimage $\psi^{-1}(\psi(H)) = H \cdot Q$ is closed in G. □

6 Free products of GFERF groups

In the previous section we considered hyperbolic groups which engulf every proper quasiconvex subgroup. For brevity we call them *QE-groups*.

We see from Theorem 2 that any QE-group G is close to being GFERF. In fact, G is quasi-isometric to the quotient $G/E(G)$ which is GFERF by Corollary 3 and Theorem 3. Nevertheless, it is still unclear whether each QE-group is GFERF. Theorem 2 would yield a positive answer if a free product of any two QE-groups were itself a QE-group. Unfortunately, the author is unable to prove this; in fact, he doubts if it is true in general. However, the following statement, proved by Burns, can be used to show that a free product of GFERF groups is again GFERF:

Lemma 6.1 ([2, Theorem 1.1]). *Suppose that G is the free product of a family of subgroups G_i of G indexed by some set I, and let H be a finitely generated subgroup. If for all $i \in I$, $g \in G$, the subgroup $H^g \cap G_i$ is G_i-separable, then H is G-separable.*

Let H be a subgroup of a group G. Suppose that $\mathscr{A}$ and $\mathscr{B}$ are finite generating sets for G and H respectively and $|\cdot|_G$, $|\cdot|_H$ are the corresponding length functions. Set $\hat{c} = \max\{|b|_G : b \in \mathscr{B}\}$. Evidently $|h|_G \leqslant \hat{c}|h|_H$ for all $h \in H$.

We say that H is *undistorted* in G if there exists a constant $c \geqslant 0$ such that $|h|_H \leqslant c|h|_G$ for every $h \in H$. In a hyperbolic group G, a finitely generated subgroup is undistorted if and only if it is quasiconvex (see [10, Lemma 1.6]).

Proof of Theorem 4. It is well known that a free product of hyperbolic groups is a hyperbolic group (see, for instance, [4, (1.34)]). Thus G is hyperbolic. Clearly, the subgroups G_1 and G_2 are undistorted in G; consequently, they are quasiconvex.

Choose an arbitrary quasiconvex subgroup $H \leqslant G$, an element $g \in G$ and $i \in \{1, 2\}$. The subgroup H^g is quasiconvex by Remark 2.1. Since the intersection of two quasiconvex subgroups is quasiconvex, from [21, Proposition 3], the subgroup $H^g \cap G_i$ is quasiconvex in G. Consequently, $H^g \cap G_i$ is undistorted in G, and hence undistorted in G_i. Thus $(H^g \cap G_i)$ is G_i-separable because G_i is GFERF. By Lemma 6.1, H is G-separable. □

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